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C^r-RIGIDITY THEOREMS FOR HYPERBOLIC FLOWS

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ABSTRACT

We consider the differentiability of a conjugating homeomorphism for co-dimension-one hyperbolic flows, under certain measureability conditions. The simple central idea is to use symbolic dynamics to apply the analysis for the simpler case of internal maps.

§1. Introduction

A central problem in the study of hyperbolic systems is to classify them into classes with common dynamical properties. From the viewpoint of ergodic theory it is natural to look for a measure-preserving isomorphism which conjugates the dynamics of the two systems. When studying the topological dynamics it is more appropriate to require the conjugating map to be a homeomorphsim and so to classify up to topological conjugacy. For higher degrees of differentiability in the two systems we can try to classify them up to conjugating maps which have appropriate degrees of differentiability.

In this paper we shall investigate the relationship between these different types of conjugacy, particularly in the context of Anosov flows.

The idea of a C^r-rigidity theorem has been promoted in [14] and [5]. The basic question is: When can a measure isomorphism or topological conjugacy be 'lifted' to a C^r conjugacy? ($r \geqq 1$), i.e. the conjugating map should have a C^r-version.

Feldman and Ornstein have a form of 'dynamical rigidity theorem' which is applicable to geodesic flows on the unit tangent bundle of surfaces of negative

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curvature: Let $\phi_t : M \to M$ and $\chi_t : M' \to M'$ be 3-dimensional C^2-Anosov flows which preserve smooth measures and have C^1 non-integrable horocycle foliations, then any conjugating homeomorphism $h : M \to M'$ is necessarily a C^1-diffeomorphism [5].

In this paper we shall consider several extensions of this result. This will include the cases where there exists no smooth invariant measure, or the foliations are of class $C^{r+\alpha}$ ($r \geqq 1$), or the flows are assumed to be co-dimension one Anosov flows.

In Section 2 we develop the theme of C^r-rigidity theorems, after Shub and Sullivan [14], in the setting of piecewise C^r expanding Markov interval maps. We recover their results for $r \geqq 2$ as well as proving a more general Markov version.

In Section 3 we introduce the symbolic dynamics for Anosov flows as developed and refined by Ratner and others [11], [12]. This is important in applying the preceding results for interval maps to Anosov flows.

In Section 4 we study C^r-rigidity theorems for codimension one Anosov flows.

In the final section, Section 5, we restrict our interest to C^r-rigidity theorems for 3-dimensional Anosov flows and prove our main results.

The author's interest in these problems was motivated by the papers of Shub–Sullivan [14] and Feldman–Ornstein [5] and by lectures given by Dennis Sullivan at IHES (October–December 1984).

I am grateful to Keith Burns for making the Feldman–Ornstein paper available to me and also to Howard Sealey for some discussions on its content. I would like to express my gratitude to the referee for many enlightening comments and suggestions.

§2. Markov interval maps

In this section we shall be concerned with Markov interval maps and the relationships between their equivalence classes when classified up to topological conjugacy, C^r-conjugacy, and (Lebesgue) measure-preserving conjugacy.

The starting point for this analysis is the following result of Shub and Sullivan [14]: If $f, g : S^1 \to S^1$ are a pair of strictly expanding C^r maps (where $2 \leqq r \leqq \omega$) then any absolutly continuous map $h : S^1 \to S^1$ satisfying $hf = gh$ must have a C^r-version. (Here an absolutely continuous map $h : S^1 \to S^1$ is one for which sets of positive Lebesgue measure are carried to set of positive Lebesgue measure.)

We shall consider interval maps which admit discontinuities, providing they lie in a finite invariant set. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a finite set of disjoint points on the real line. Let f: $[0, 1] \to [0, 1]$ be a map which is uniquely defined on C^r, except possibly at the points $x_0, \ldots, x_n$. In addition, we assume that for each x_i each of the limits $\lim_{x \searrow x_i} f(x)$, $\lim_{x \nearrow x_i} f(x)$ lies in the finite set $\{x_0, \ldots, x_n\}$ (for x_0 and x_n only one such limit is considered). We call interval maps which have this property *Markov maps*.

We shall from now onwards assume that f is *expanding*. That is, there exists $\beta > 1$ such that $|f'(x)| \geqq \beta$ for all $0 \leqq x \leqq 1$. (At $x_0, \ldots, x_n$ we may get more than one value for f' by taking derivatives from the left and from the right.)

One advantage of expanding Markov maps is that they give a simple and concise way of studying orbits under iterations of the map f: $[0, 1] \to [0, 1]$. Given a point $w \in [0, 1]$ the orbit $w, fw, f^2w, \ldots$ determines a sequence $w_1, w_2, w_3, \ldots \in \{0, 1, \ldots, n-1\}$ where $x_{w_i} \leqq f^i w \leqq x_{w_i+1}$. The sequence $\boldsymbol{w} = (w_i)_{i=0}^{\infty}$ associated with w will be uniquely defined, except in the case where the orbit of w eventually lands in the set $\{x_1, \ldots, x_n\}$.

We can conveniently collect together these sequences as follows. Let A be the $n \times n$ matrix whose entries are either zero or unity according to the following prescription:

$$A(i,j) = \begin{cases} 1 & \text{if } f(x_i, x_{i+1}) \supseteq (x_j, x_{j+1}), \\ 0 & \text{if } f(x_i, x_{i+1}) \cap (x_j, x_{j+1}) = \varnothing. \end{cases}$$

(Our assumptions imply that these are the only possibilities.) Let $\Sigma_A = \{\boldsymbol{w} \in \Pi_0^{\infty} \{0, \ldots, n-1\} \mid A(w_i, w_{i+1}) = 1\}$. Then we can define a metric on this space by $d(\boldsymbol{y}, \boldsymbol{z}) = \Sigma_{n=0}^{\infty} e(y_n, z_n)/\beta^n$ where

$$e(i,j) = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

The *shift* $\sigma: \Sigma_A \to \Sigma_A$ is given by $(\sigma x)_n = x_{n+1}$. With respect to the above metric σ becomes a local homeomorphism.

We define a map $\pi: \Sigma_A \to [0, 1]$ by $\pi(\boldsymbol{w}) = \bigcap_{i=0}^{\infty} f^{-i}[x_{w_i}, x_{w_i+1}]$. Then π satisfies the semi-conjugacy relation $\pi\sigma = f\pi$ and is Lipschitz (by the use of β in the definition of the metric).

Many different expanding maps will give rise to the same matrix A. Let us assume that g is another such map corresponding to $0 < y_1 < \cdots < y_n = 1$.

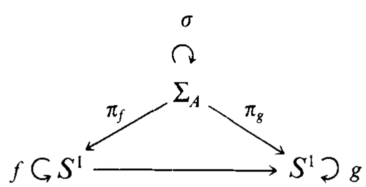
We shall also assume that f and g are consistent in preserving (or reversing) the orientations of $[x_i, x_{i+1}]$ and $[y_i, y_{i+1}]$ respectively, for $i = 0, 1, \ldots, n-1$. We then call f and g *consistent realisations* of A.

Let π_f and $\pi_g : \Sigma_A \to S^1$ be the associated semi-conjugacy maps for f and g, respectively. These induce a piecewise continuous map $h : S^1 \to S^1$ with $h\pi_f = \pi_g$.

PROPOSITION 1. (a) *Let f and g be $C^{1+\alpha}$ consistent realisations of A. An absolutely continuous map $h : S^1 \to S^1$ such that $hf = fg$ must necessarily be piecewise C^1. Even stronger*:

(b) *Let f and g be $C^{r+\alpha}$ consistent realisations of A ($r \geqq 1$). An absolutely continuous map $h : S^1 \to S^1$ such that $hf = gh$ must necessarily be piecewise $C^{r+\alpha}$*

PROOF. (a) With h defined as before we get the following commuting diagram:



We shall now freely use results from the thermodynamic formalism of $\sigma : \Sigma_A \to \Sigma_A$ (cf. [10] and [4] for details).

The map $\log|f'| \circ \pi_f : \Sigma_A \to \mathbf{R}$ is α-Hölder continuous by assumption. Accordingly, it has a unique equilibrium state μ_f on Σ_A (i.e. there exists a continuous function $u : \Sigma_A \to \mathbf{R}$ such that $d\mu_f f / d\mu_f = |f'| \circ \pi_f \exp(u\sigma - u)$). Furthermore, the measure $\nu_f = \pi_f^* \mu_f$ on S^1 is an ergodic f-invariant measure equivalent to Lebesgue measure. (This is essentially given in [3].) Similarly, if μ_g is the unique equilibrium state for $\log|g'| \circ \pi_g$ then $\pi_g^* \mu_g = \nu_g$ is an ergodic g-invariant measure equivalent to Lebesgue measure.

As a preliminary step we observe that since $h * \nu_f$ and ν_g are both ergodic and equivalent to Lebesgue measure (and thus each other) we have that $h * \nu_f = \nu_g$. Thus h is, in fact, a measure preserving conjugacy. Since we can write

$$\frac{dh * l}{dl} = \frac{dh * l}{dh * \nu_f} \cdot \frac{dh * \nu_f}{d\nu_g} \cdot \frac{d\nu_g}{dl} \qquad \text{a.e.}$$

and all terms on the right hand side of this equality make sense, we can take $h'(x)$, the derivative of h, to be defined a.e. The following step is similar to [3] p. 18.

The conjugacy $hf = gh$ gives that

$$(\log|h'|)\circ f - \log|h'| = (\log|g'|)\circ h - \log|f'| \qquad \text{a.e.}$$

Since $(\log|g'|)\circ \pi_g$ and $(\log|f'|)\circ \pi_f$ share the same equilibrium state $\mu_f = \mu_g$, we can deduce that

$$(\log|g'|)\circ \pi_g - (\log|f'|)\circ \pi_f = u\sigma - u$$

where $u : \Sigma_A \to \mathbf{R}$ is continuous. However, because of ergodicity the equality

$$[(\log|h'|)\circ \pi_f - u]\circ \sigma = (\log|h'|)\circ \pi_f - u$$

gives that $(\log|h'|)\circ \pi_f - u$ is constant. In particular $u = (\log|h'|)\circ \pi_f$ has a continuous version. The induced map $\hat{u} : S^1 \to \mathbf{R}$ given by $\hat{u}\circ \pi_f = u$ is piecewise continuous. From this we can conclude that h is piecewise C^1, as required.

(b) In the event that f and g are piecewise $C^{r+\alpha}$ ($r \geqq 1$), Markov, strictly expanding maps it is possible to follow the Shub–Sullivan proof to deduce that if the conjugating homeomorphism is absolutely continuous, then it is piecewise $C^{r+\alpha}$. The main point is that we can change f and g so that $\mu_f = l = \mu_g$, where l is Lebesgue measure. In order to do this we replace f by $T^{-1}\circ f \circ T$ where $T(x) = \int d\mu_f/dl(z)dz$. We now show that $d\mu_f/dl > 0$ can be shown to be of class $C^{r-1+\alpha}$. Recall that $d\mu_f/dl$ is the positive (maximal) eigenfunction for the Perron–Frobenius operator $L : B \to B$ given by $Lg(x) = \Sigma_{fy=x}\, g(y)/|f'(y)|$ where B is the space of piecewise $C^{r-1+\alpha}$ functions which is a Banach space with norm

$$\| h \| = \| h \|_\infty + \| h' \|_\infty + \cdots + \| h^{(r-1)} \|_\infty + \sup_{x\neq y}\left[\frac{|h^{(r-1)}(x) - h^{(r-1)}(y)|}{d(x,y)^\alpha}\right].$$

We now have that f, T (and also T^{-1}) are $C^{r+\alpha}$. It follows that $T^{-1}\circ f\circ T$ is piecewise $C^{r+\alpha}$ and also Lebesgue measure is the invariant measure for $T^{-1}\circ f\circ T$.

Similarly we can choose a $C^{r+\alpha}$ map $S : S^1 \to S^1$ such that $S^{-1}\circ g\circ S$ is piecewise $C^{r+\alpha}$ and Lebesgue measure is invariant. From this we can deduce that the corresponding conjugacy $S^{-1}\circ h\circ T$ is a piecewise linear interval exchange map, hence h is $C^{r+\alpha}$.

REMARK. The consistency condition has a pleasant, if modest, topological formulation. Let M be the quotient space formed from S^1 by identifying the n

discontinuity points $x_1, \ldots, x_n$. The fundamental group of M is then $\pi_1(M) \simeq \mathbf{Z}^n$. We let $e_1, \ldots, e_n$ denote the generators. The map $f: M \to M$ induces a transformation $f^*: \pi_1(M) \to \pi_1(M)$ which can be written as a matrix $B = B(f)$ with entries 0, ± 1. Then the condition that f and g are consistent realisations is equivalent to $B(f) = B(g)$.

§3. Hyperbolic flows and symbolic dynamics

In his thesis on geodesic flows for manifolds of negative sectional curvature Anosov showed that many of the features of these flows were preserved for a broader class of flows satisfying a few simple axioms [1]. (Such flows were subsequently termed Anosov flows.)

To make the connection between Anosov flows and interval maps we must appeal to symbolic dynamics and the construction of Markov partitions.

Let M be a compact manifold and let $\phi_t : M \to M$ be C^∞. We say that ϕ is *Anosov* if $TM = E^0 \oplus E^s \oplus E^u$ where E^0, E^s and E^u are continuous, $D\phi_t$-invariant bundles such that:

(a) E^0 is the one-dimensional bundle tangent to the flow direction;

(b) there exist $C, \lambda > 0$ such that

$$\left.\begin{aligned} \| D\phi_t(v) \| &\leqq Ce^{-\lambda t} \| v \| \quad \text{for } v \in E^s, \\ \| D\phi_{-t}(v) \| &\leqq Ce^{-\lambda t} \| v \| \quad \text{for } v \in E^u. \end{aligned}\right\}$$

For $x \in M$ the sets $W^s(x) = \{y \mid d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to +\infty\}$, $W^u(x) = \{y \mid d(\phi_{-t} x, \phi_{-t} y) \to 0 \text{ as } t \to +\infty\}$ form C^β-immersed submanifolds of dimensions k and l, respectively, where $k = \dim E^s$ and $l = \dim E^u$ [9].

The stable manifolds form the k-dimensional C^∞ leaves of a foliation $\mathscr{F}^s$. Similarly, the unstable manifolds form l-dimensional C^∞-leaves of a foliation $\mathscr{F}^u$.

We recall that a foliationn $\mathscr{F}$ of an n-dimensional manifold M with p-dimensional leaves is called C^q if there exists a family of C^q charts $\psi: U \to \mathbf{R}^n$ such that leaves of the foliation $\mathscr{F}$ (restricted to U) are carried to hyperplanes of the form $\mathbf{R}^p \times \{y\}$ (restricted to $\psi(U)$).

Unfortunately, although the leaves of $\mathscr{F}^s$ and $\mathscr{F}^u$ are C^∞ it is only known, in general, that the foliations are continuous (i.e. C^0). However, in the case of geodesic flows associated with locally symmetric compact manifolds of negative curvature the foliations are known to be real-analytic (i.e. C^ω) [7].

There is a third foliation $\mathscr{F}^0$ of M for which the one-dimensional leaves of

$\mathscr{F}^0$ are simply the trajectories of points in M under ϕ. The foliation $\mathscr{F}^0$ is of class C^∞.

Following Ratner [11], [12] we can construct Markov sections for the flow.

Let $T_0, \ldots, T_{n-1}$ be transverse $(k+l)$-dimensional sections transverse to the flow direction. The basic idea is that we can keep track of the orbit of a point $x \in M$ by the sections it crosses. Thus for $x \in M$ we assume that $\phi_t x$ $(t > 0)$ intersects the sections $T_{x_1}, T_{x_2}, T_{x_3}, \ldots$. Similarly we can assume that $\phi_{-t} x$ intersects $T_{x_0}, T_{x_{-1}}, T_{x_{-2}}, \ldots$ (for $t > 0$). In particular the sequence $\boldsymbol{x} = (x_n)_{-\infty}^{+\infty}$ tells us which sections the orbit of x passes through, in order, and that x lies on its orbits segment between T_{x_0} and T_{x_1}. The hyperbolicity assumption for Ansosov flows plays an important role here in that it allows the sections $T_0, \ldots, T_{n-1}$ to be chosen so that the totality of sequences allowed forms a sequence space determined by an $n \times n$ matrix A with 0–1 entries. This space is denoted X_A and defined by

$$X_A = \left\{ \boldsymbol{x} \in \prod_{-\infty}^{+\infty} \{0, \ldots, n-1\} \,\middle|\, A(x_i, x_{i+1}) = 1 \right\}.$$

(There is a slight difficulty when the orbit of a point passes through the 'boundary' of a section. In this event the orbit still determines only a finite number of sequences in a self-consistent way, cf. [12] for details.)

We define a shift $\sigma : X_A \to X_A$ by $(\sigma x)_n = x_{n+1}$ and put a metric d on X_A by

$$d(x, y) = \sum_{n=-\infty}^{+\infty} e(x_n, y_n)/2^{|n|}$$

where

$$e(i,j) = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

With respect to this metric X_A is compact and zero-dimensional and σ becomes a homeomorphism.

A sequence $\boldsymbol{x} \in X_A$ corresponds to an orbit segment in M and, in particular, to a point in T_{x_0} which we denote $\pi(x)$. This defines a Hölder continuous map $\pi : X_A \to \bigcup_{i=0}^{n-1} T_i$. Let $r(\boldsymbol{x})$ be the length of the orbit segment between $\pi(\boldsymbol{x})$ and $\pi(\sigma \boldsymbol{x})$, then $r : X_A \to \mathbf{R}^r$ is a positive Hölder continuous function.

Let $X_A^r = \{(\boldsymbol{x}, s) \in X_A \times \mathbf{R}^r \mid 0 \leqq s \leqq r(x)\}$ where the points $(\boldsymbol{x}, r(\boldsymbol{x}))$ and $(\sigma x, 0)$ are identified. We define the *symbolic flow* $\sigma_t^r : X_A^r \to X_A^r$ by $\sigma_t^r(\boldsymbol{x}, s) =$

$(\boldsymbol{x}, s+t)$, where we make use of the identifications where appropriate. This can be thought of as flowing vertically under the graph of r. The map $\pi : X_A \to \bigcup_i T_i$ can be extended to $\pi : X_A^r \to M$ by simply defining $\pi(\boldsymbol{x}, t) = \phi_t \pi(\boldsymbol{x})$, where $\boldsymbol{x} \in X_A$. A basic property of this construction is that $\pi : X_A^r \to M$ satisfies $\pi \sigma_t^r = \phi_t \pi$, for all $t \in \mathbf{R}$, where π is a continuous, bounded-to-one, surjective semi-conjugacy [12].

In many problems about Anosov flows it is advantageous to 'lift' the problem to the symbolic flow where it may be more accessible (frequently through the use of techniques from thermodynamics).

We should say a little more about the construction of the transverse sections $T_1, \ldots, T_n$. For each section T_i there exists a point $z_i \in T_i$ for which a neighbourhood of z_i in the unstable manifold $W^u(z_i)$ forms a k-dimensional submanifold of T_i. That is, locally $W^u(z_i)$ is contained in T_i. Furthermore, if $y \in W^u(z_i) \cap T_i$ lies in the same path connected component as z_i, then locally $W^s(y)$ is a l-dimensional submanifold of T_i. That is, locally $W^s(y)$ is contained in T_i. Thus T_i can be thought of as the union of sections of $W^s(y)$ which are 'ribs' held together by a 'spine' consisting of a section of $W^u(z_i)$.

If $\boldsymbol{w} \in X_A$ satisfies that $\pi(\boldsymbol{w}) = z_i$, then $W^u(z_i)$ restricted to T^i is the image under π of the set $\{y \in X_A \mid y_k = w_k, k \leqq 0\}$. Let $W^u(z_i, T_i)$ denote the segment of $W^u(z^i)$ restricted to T_i. We denote by $H : \bigcup_i T_i \to \bigcup_i T_i$ the Poincaré map induced by ϕ_t, $t > 0$. (More precisely $H\pi(\boldsymbol{x}) = \phi_{r(\boldsymbol{x})}\pi(\boldsymbol{x})$.)

The reason that orbits in M correspond to sequences in X_A is that the sections $T_0, \ldots, T_{n-1}$ can be chosen to satisfy a certain Markovian property: $H(W^u(z_i, T_i)) \cap T_j$ (where $A(i,j) = 1$) corresponds to a component of the image of an unstable manifold projected onto T_j along flow lines (cf. [11], [12]). There is a canonical map $P_j : T_j \to W^u(z_j)$ by projecting along the segments of stable manifolds whose union is T_j. Thus composing H with P_j gives us a map $f_j : W^u(z_i, T_i) \cap H^{-1}T_j \to W^u(z_j, T_j)$ when $A(i,j) = 1$.

For Anosov flows in general it is not always the case that they preserve a smooth measure. However, for any Anosov flow we can construct the *Sinai–Ruelle–Bowen measures* μ^+ and μ^- on M which project to smooth meaures on the unstable and stable manifolds respectively [4]. A necessary and sufficient condition for μ to be a smooth ϕ-invariant measure is that $\mu = \mu^+ = \mu^-$.

We remark that any C^1 diffeomorphism $h : M \to M'$ which conjugates the flows must necessarily take the SRB-measures for ϕ onto the corresponding measures for ψ [4]. Thus, in particular, if ϕ and ψ preserve smooth measures then h carries one measure to the other.

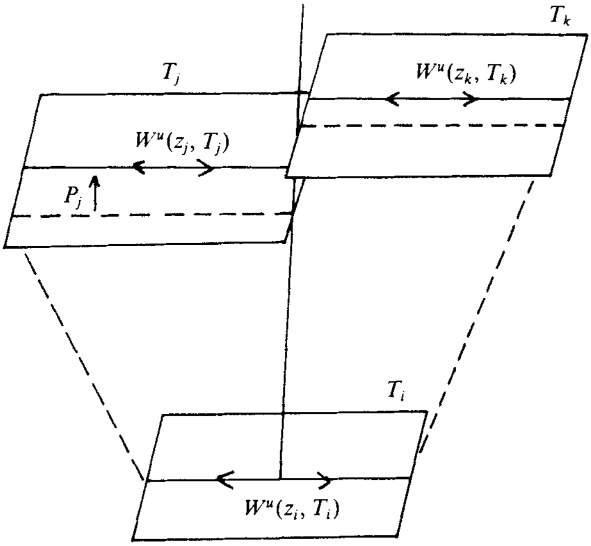


Fig. 1

§4. Condimension-1 Anosov flows

We shall now consider how the preceding analysis relates to the study of codimension-1 Anosov flows. These are Anosov flows for which either $\dim E^s = 1$ or $\dim E^u = 1$. In particular, Anosov flows on any manifold of dimension either 3 or 4 are automatically codimension-1 Anosov flows. For definiteness we shall assume that $\dim E^u = 1$. (The case of $\dim E^s = 1$ is easily recovered by reversing the flow under $t \mapsto -t$.)

We shall assume that the unstable foliation $\mathscr{F}^u$ is of class $C^{1+\alpha}$. The importance of this lies in that the maps f_j on segments of (one-dimensional) unstable manifolds are of class $C^{1+\alpha}$. In particular, we can interpret the finite family of maps f_j as a single $C^{1+\alpha}$ expanding Markov map of the interval (since P_j, as defined in Section 3, in $C^{r+\alpha}$).

Assume that $\phi_t : M \to M$ and $\psi_t : M' \to M'$ are Anosov flows and that ϕ has codimension-1 stable manifolds (one-dimensional unstable manifolds). Let $h : M \to M'$ be a homeomorphism which conjugates ϕ and ψ. From the definitions we see that h preserves stable and unstable manifolds (and also that ψ is also a codimension-1 flow).

The transverse sections $T_0, \ldots, T_{n-1}$ constructed for ϕ are carried to transverse sections $S_j = h(T_j), j = 0, \ldots, n-1$ for ψ. Thus the flow $\psi_t : M' \to M'$ generates an interval map g in terms of the Poincaré map between sections.

Such is the construction that we have interval maps $f: I \to I$ and $g: J \to J$ and a homeomorphism $H: I \to J$ such that $hf = gh$. (To be consistent with our earlier notation we should rescale to make I and J unit intervals.)

Let μ^+ be the SRB measure for ϕ_t $(t > 0)$ and let v^+ be the corresponding SRB-measure for ψ_t $(t > 0)$. These measures give rise to measures equivalent to Lebesgue measure on unstable manifolds. If we assume $h: M \to M'$ is a homeomorphism which conjugates the flows and is absolutely continuous with respect to the SRB measures, then this induces a homeomorphism $h: I \to J$ conjugating f and g and absolutely continuous with respect to Lebesgue measure. We know by Proposition 1(a) that $h: I \to J$ is necessarily C^1. We can thus conclude that $h: M \to M'$ is necessarily C^1 (except possibly at points corresponding to the boundaries of sections) on unstable fibres. The difficulty on the boundaries of Markov sections are irrelevant, since they are an artifact of the construction and because the sections are constructed somewhat arbitrarily. In particular, any prechosen point can be made to be interior to the sections. If $\mathscr{F}^u$ is $C^{r+\alpha}$, then we can assume $h: I \to J$ is $C^{r+\alpha}$ and proceed as above to deduce that $h: M \to M'$ is $C^{r+\alpha}$.

We summarize our conclusions below.

PROPOSITION 2. (a) *Let $\phi_t: M \to M$ and $\psi_t: M' \to M'$ be codimension-*1 *Anosov flows (with* $\dim E^u = 1$) *and $C^{1+\alpha}$ foliations $\mathscr{F}^u$. Let $h: M \to M'$ be a homeomorphism which conjugates the flows and such that h is absolutely continuous with respect to the SRB-measure μ^+, v^+. Then h is C^1 on unstable manifolds. Even stronger:*

(b) *Let $\phi_t: M \to M$ and $\psi_t: M' \to M'$ be Anosov flows with* $\dim E^u = 1$ *and $C^{r+\alpha}$ foliations $\mathscr{F}^u$ $(r \geqq 1)$. Let $h: M \to M'$ be a homeomorphism which conjugates the flows and such that h is absolutely continuous with respect to the SRB-measure μ^+ and v^+. Then h is $C^{r+\alpha}$ on unstable manifolds.*

REMARK. We have chosen to confine ourselves wholly to the situation for flows. We should point out that somewhat analogous results are true for codimension-1 Anosov diffeomorphism.

We give these below.

(i) given two Anosov diffeomorphisms with $\dim E^u = 1$ (and $C^{1+\varepsilon}$ foliations by unstable manifolds) conjugated by a homeomorphism which is absolutely continuous with respect to the (positive) SRB-measures, then the homeomorphism is necessarily C^1 on unstable fibres.

(ii) Given two Anosov diffeomorphisms $(r \geqq 1)$ with $\dim E^u = 1$ and $C^{r+\varepsilon}$ foliations by unstable manifolds, then any homeomorphism which conjugates the diffeomorphisms and is absolutely continuous with respect to the SRB-measures is necessarily $C^{r+\varepsilon}$ on the unstable fibres.

§5. Three-dimensional Anosov flows

In the previous section we considered codimenion-1 Anosov flows. The situation where $\dim M = 3$ deserves separate attention, and is the subject of this section. When the manifold is 3-dimensional we are in the situation where $\dim E^u = 1 = \dim E^s$ and we can apply Proposition 2 to the stable and unstable manifolds simultaneously and hence deduce information on the differentiability of h on the entire manifold, rather than restricted to fibres of $\mathscr{F}^u$ and $\mathscr{F}^s$.

For 3-dimensional Anosov flows we can apply Proposition 2 to deduce that h is $C^{1+\alpha}$ along both the one-dimensional strong stable and strong unstable manifolds. Together with the orbit foliation these form a $C^{1+\alpha}$ co-ordinate system. Hence we have the following.

PROPOSITION 3. *Let $\phi_t : M \to M$ and $\psi_t : M' \to M'$ be three-dimensional Anosov flows with $C^{1+\alpha}$ stable and unstable foliations. Let $h : M \to M'$ be a conjugating homeomorphism which is absolutely continuous with respect to each of the positive and negative SRB-measures. Then h is necessarily a C^1-diffeomorphism.*

A special case of the above theorem is where each flow has a smooth invariant measure (which is necessarily equal to both the positive and negative SRB-measures). In this context the above theorem would give that h is C^1 if and only if h is absolutely continuous with respect to the Riemannian measures.

Feldman and Ornstein have shown that under the hypothesis that: the flows are C^2; the flows preserve smooth measures; the foliations are C^1; and the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ are not integrable, then *any* homeomorphism which conjugates the two flows is necessarily C^1 [5] (and, in particular, preserves the smooth measure). Their proof makes essential use of the non-integrability condition (in itself an innocuous assumption in view of the results of Plante [10]).

THEOREM. (a) *Let $\phi_t : M \to M$ and $\psi_t : M' \to M'$ be 3-dimensional Anosov flows with $C^{r+\alpha}$-foliations ($r \geqq 1$) and assume $h : M \to M'$ is a conjugating homeomorphism which is absolutely continuous with respect to each of the* *positive and negative SRB-measures. Then h is necessarily a $C^{r+\alpha}$-diffeomorphism.*

(b) *Let $\phi_t: M \to M$ and $\psi_t: M' \to M'$ be 3-dimensional Anosov flows with smooth invariant measures and $C^{r+\alpha}$ non-integrable unstable and stable foliations. Let $h: M \to M'$ be a conjugating homeomorphism, then h is necessarily a $C^{r+\alpha}$-diffeomorphism.*

PROOF. (a) This is a direct consequence of Proposition 2(b).

(b) By the Feldman–Ornstein result h must be a C^1-diffeormorphism and consequently preserve smooth measures. We can then 'enrich' the differentiability class of h to $C^{r+\alpha}$ by applying part (a).

EXAMPLES. (I) Feldman and Ornstein showed that the topological conjugacy between geodesic flows for compact surfaces of strictly negative curvature must necessarily be of class C^1. From the theorem above it becomes clear that this conjugacy is as differentiable as the foliations. Hurder and Katok have established that the foliations are $C^{1+\alpha}$, for any $0 \leqq \alpha < 1$, and so we can slightly sharpen the Feldman–Ornstein result to say the conjugacy is $C^{1+\alpha}$.

(II) For compact surfaces of *constant* negative curvature the foliations for the associated geodesic flow are known to be real analytic. It is a fact, already known to people in the field, that any conjugating homeomorphism h between two such flows is algebraic. Another proof is to use our theorem to first deduce that it is analytic. Since the unit tangent bundles have PSL(2, $\mathbf{R}$) as universal covers, the lift $\hat{h}$: PSL(2, $\mathbf{R}$) $\to$ PSL(2, $\mathbf{R}$) of h must be algebraic.

(III) In a recent paper E. Ghys constructs examples of 'exotic' 3-dimensional Anosov flows for which the foliations are all smooth but *not* analytic [6]. It is observed there (Remarque 4.8) that any conjugating homeomorphism between two such flows in smooth. This also follows immediately from our theorem.

REMARK. We can, of course, derive discrete versions of these results for Anosov diffeomorphisms:

(i) Let $f: M \to M$ and $g: M' \to M'$ be $C^{1+\alpha}$ two-dimensional Anosov diffeomorphisms (with $C^{1+\alpha}$ stable and unstable foliations). Let $h: M \to M'$ be a conjugating homeomorphism which is absolutely continuous with respect to each of the SRB-measures (both positive and negative). Then h is necessarily a C^1-diffeormorphism.

(ii) Let $f: M \to M$ and $g: M' \to M'$ be two-dimensional Anosov diffeomorphisms with $C^{r+\alpha}$ stable and unstable foliations ($r \geqq 1$). Let $h: M \to M'$ be a conjugating homeomorphism which is absolutely continuous with respect to each of the positive and negative SRB-measures. Then h is necessarily a $C^{r+\alpha}$ diffeomorphism.

Appendix: Cantor sets and Hausdorff dimension

In the previous formulation the maps $f: S^1 \to S^1$ were surjective and Lebesgue measure was the appropriate measure to study. We now want to consider a Cantor set in the interval which is invariant under an expanding transformation. In particular, we want to study the different types for conjugating maps which can occur between two such systems.

Let $I_1, \ldots, I_n$ be disjoint closed intervals in S^1. Let $f: \bigcup_{i=1} I_i \to S^1$ be a C^r-map on each of the intervals I_i which is strictly expanding, i.e. $|f'(x)| \geqq \beta > 1$ for all $x \in \bigcup_{i=1}^n I_i$ and where end points are mapped outside the union of intervals. Let Λ be the limit set $\Lambda = \bigcap_{j=0}^{\infty} f^{-j}(\bigcup_{i=1}^n I_i)$. Assume that Λ is a Cantor set.

There is an accepted notion of 'size' of subsets of the interval called the Hausdorff dimension. Since Λ is a compact metric space we can define the Hausdorff dimension S as follows: For $\rho > 0$ and a countable open cover $\mathscr{U}$ of Λ let $d(\rho, \mathscr{U}) = \Sigma_{U \in \mathscr{U}} (\operatorname{diam} U)^{\rho}$. For any $\varepsilon > 0$ we let $d_{\varepsilon}(\rho) = \inf_{\mathscr{U}} d(\rho, \mathscr{U})$, where the infimum is taken over all countable covers for which diam $U \leqq \varepsilon$ for each $U \in \mathscr{U}$, and let $d(\rho) = \sup_{\varepsilon > 0} d_{\varepsilon}(\rho)$. There exists a unique $0 \leqq \delta \leqq 1$ such that $d(\rho) = 0$ for $\rho > \delta$ and $d(\rho) = +\infty$ for $\rho < \delta$. The value δ is defined to be the *Hausdorff dimension* of Λ.

A set of positive Lebesgue measure has Hausdorff dimension 1 and a countable set has Hausdorff dimension 0. In the above context it can be shown that $0 < \delta < 1$.

Associated with Λ and δ is a canonical measure μ, supported on Λ, called the *Hausdorff measure* and defined by

$$\frac{d\mu f}{d\mu} = |f'|^{\delta},$$

Assume that Λ a cantor set, i.e. uncountable, closed and nowhere dense. Assume $f(I_i) \cap I_j \neq \varnothing \Rightarrow I_j \subset f(I_i)$. Then it follows that f is conjugate to $\sigma: \Sigma_A \to \Sigma_A$ for an appropriate Λ. If f, g are two such maps which are both C^r and give rise to the same matrix A, then we shall again use the term *consistent realisations* to describe them.

Assume $(\Lambda_f, \mu_f, \delta_f)$ and $(\Lambda_g, \mu_g, \delta_g)$ are the limit sets and Hausdorff measures and dimensions associated with f and g, respectively. There is a canonical map from Λ_f to Λ_g taking points given by intersections of intervals $\bigcap_{j=0}^{\infty} f^{-n}(I_{i_j})$ to points $\bigcap_{j=0}^{\infty} g^{-j}(J_{i_j})$ where $(i_j)_{j=0}^{+\infty} \in \Sigma_A$. If $h : \Lambda_f \to \Lambda_g$ is the canonical map and $h * \mu_f = \mu_g$, then by the definition of Hausdorff measure we see that

$$|f'|^{\delta_f} = \frac{d\nu_f f}{d\nu_f} = \frac{d\nu_g g}{d\nu_g} \circ h = |g'|^{\delta_g} \circ h.$$

If $g'' \neq 0$ on some neighbourhood U, then by the implicit function theorem $h|_{h^{-1}U}$ is C^{r-1}. Given $x \in \Lambda_f$, choose n such that $f^n(h^{-1}U) \ni x$, then in a neighbourhood of x we have that $h(y) = f^n \circ h \circ g^{-n}(y)$ is well defined and C^{r-1}. Thus h is C^{r-1}. This approach will only fail if g is piecewise linear. If g is piecewise linear and f is not, then we simply interchange them. If f and g are both piecewise linear, then the result follows simply. This is close in spirit to the Sullivan–Shub argument [14]. We summarise below.

PROPOSITION 4. *Let f, g be C^r-consistent realisations for A where $h : \Lambda_f \to \Lambda_g$ preserves the Hausdorff measure. Then h extends locally to a conjugacy of class C^{r-1}.*

APPLICATION. We recall Bowen's result that all one-dimensional Axiom A flows (restricted to the non-wandering set) are homeomorphic to suspended flows [2]. Assume that such an Axiom A flow occurs in a 3-dimensional manifold. Choose transverse sections to the flow and consider the Hausdorff measure of the intersection with the non-wandering set. A conjugating homeomorphism between two such C^r flows will have a C^{r-1}-extension if and only if h preserves the Hausdorff measure on sections. (This appeals to Proposition 4.)

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